ASIL SHAAR (PROBABILITY THEORY(STAT3321))

CHAPTER 4

Chapter 4

Distributions of functions of random variables

4.1 Sampling Theory

Def: A statistic is function of random variables.

$$T = \psi(X_1, \ldots, X_n)$$

Ex. $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{X_1 + X_2 + \cdots + X_n}{n}$

$\bar{X}$ is a statistic

Def: let $X_1, \ldots, X_n$ be random variables.

① If $f_1 = f_2 = f_3 = \cdots = f_n$

we say $X_1, \ldots, X_n$ are identical

② $f(x_1, \ldots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdots f_n(x_n)$

we say $X_1, \ldots, X_n$ independent

→

③ If $X_1, \ldots, X_n$ are identical and independent

we say $X_1, \ldots, X_n$ are i.i.d. (identically indep. distributional)

or we say $X_1, \ldots, X_n$ are a random sample.

Ex X_1, X_2 random sample from a standard normal dist

what is the dist. of:

① X_1^2

② X_2^2

③ $X_1^2 + X_2^2$

1) $X_1^2 \sim \chi^2(1)$

2) $X_2^2 \sim \chi^2(1)$

3) $X_1^2 + X_2^2 \sim \chi^2(2)$

4.2 Transformation of Variables of the Discrete Type.

Theorem 1

let X be a r.v with space A

let Y be a r.v with space B

let $u : A \to B$ be such that $Y = u(X)$ is one to one Then:

$$g(y) = \begin{cases} f(w(y)) & , y \in B \\ 0 & , \text{else} \end{cases}$$

where $w = u^{-1}$ $\Rightarrow$

Theorem 2

let X_1, X_2 be random variables with space A

let Y_1, Y_2 be random variables with space B

let $u : A \to B$

such that $u(x_1, x_2) = (y_1, y_2)$

$$Y_1 = u_1(X_1, X_2)$$

$$Y_2 = u_2(X_1, X_2)$$

is one-to-one. Then:

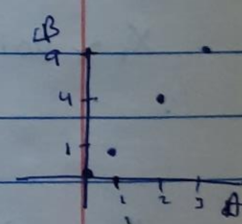
$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) & , (y_1, y_2) \in B \\ 0 & , \text{else.} \end{cases}$$

Ex 1

$\boxed{X \sim b(3, \frac{2}{3})}$ binomial

$Y = X^2$ find p.d.f of Y.

$$f(x) = \begin{cases} \binom{3}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x} & , x = 0, 1, 2, 3 \\ 0 & , \text{else} \end{cases}$$



$Y = X^2 \quad u(x) = x^2 \qquad A = \{0, 1, 2, 3\}$

$w^{-1}(y) = \sqrt{y} \qquad$ مربعه $B = \{0, 1, 4, 9\}$

$u(x)$ is one-to-one

$w(y) = u^{-1}(y) = \sqrt{y} \quad \Rightarrow$

$$g(y) = \begin{cases} f(w(y)) & , y \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} f(\sqrt{y}) & , y = 0, 1, 4, 9 \\ 0 & , \text{else.} \end{cases}$$

$$g(y) = \begin{cases} \binom{3}{\sqrt{y}} \left(\frac{2}{3}\right)^{\sqrt{y}} \left(\frac{1}{3}\right)^{3-\sqrt{y}} & , y = 0, 1, 4, 9 \\ 0 & , \text{else.} \end{cases}$$

Ex 2

X_1, X_2 independent poisson (μ_1), Poisson (μ_2) respectively.

Find the density of $X_1 + X_2$

$X_1 \sim \text{poisson}(\mu_1)$ $\quad f_1(x_1) = \begin{cases} \dfrac{e^{-\mu_1} \mu_1^{x_1}}{x_1!} & , x_1 = 0, 1, 2, \ldots \\ 0 & , \text{else} \end{cases}$

$X_2 \sim \text{Poisson}(\mu_2)$ $\quad f_2(x_2) = \begin{cases} \dfrac{e^{-\mu_2} \mu_2^{x_2}}{x_2!} & , x_2 = 0, 1, 2, \ldots \\ 0 & , \text{else.} \end{cases}$

X_1, X_2 independent $\Rightarrow f(x_1, x_2) = \begin{cases} \dfrac{e^{-\mu_1} e^{-\mu_2} \mu_1^{x_1} \mu_2^{x_2}}{x_1!\, x_2!} & , \begin{array}{l} x_1 = 0, 1, 2, \ldots \\ x_2 = 0, 1, 2, \ldots \end{array} \\ 0 & , \text{else} \end{cases}$

$\rightarrow$

$$\begin{bmatrix} y_1 = u_1(x_1, x_2) = x_1 + x_2 \\ y_2 = u_2(x_1, x_2) = x_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 = w_1(y_1, y_2) = y_1 - y_2 \\ x_2 = w_2(y_1, y_2) = y_2 \end{bmatrix}$$

$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)), & (y_1, y_2) \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} f(y_1 - y_2, y_2) & , (y_1, y_2) \in B \\ 0 & , \text{else.} \end{cases}$$

$$\boxed{g(y_1, y_2) = \begin{cases} \dfrac{e^{-(\mu_1+\mu_2)} \mu_1^{y_1-y_2} \mu_2^{y_2}}{(y_1-y_2)!\, y_2!} & , y_1 = 0, 1, 2, \ldots \; y_2 = 0, 1, 2, \ldots, y_1 \\ 0 & , \text{else} \end{cases}}$$

$$(a+b)^m = \sum_{k=0}^{m} \frac{m!}{(m-k)!\,k!} a^k b^{m-k}$$

$$g_1(y_1) = \sum_{y_2} g(y_1, y_2) = \sum_{y_2=0}^{y_1} \frac{e^{-(\mu_1+\mu_2)} \mu_1^{y_1-y_2} \mu_2^{y_2}}{(y_1-y_2)!\, y_2!}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!} \left[\sum_{y_2=0} \frac{y_1!}{(y_1-y_2)!\, y_2!} (\mu_1)^{y_1-y_2} (\mu_2)^{y_2} \right] = \frac{e^{-(\mu_1+\mu_2)} (\mu_1+\mu_2)^{y_1}}{y_1!}, y_1 = 0, 1, 2 \ldots$$

$$g_1(y_1) = \begin{cases} \dfrac{e^{-(\mu_1+\mu_2)} (\mu_1+\mu_2)^{y_1}}{y_1!} & , y_1 = 0, 1, 2, \ldots \\ 0 & , \text{else} \end{cases}$$

$Y_1 = X_1 + X_2 \sim \text{poisson}(\mu_1 + \mu_2)$

4.3 Transformations of variables of continous Type

Theorem 3

X random variable on space A

Y random variable on space B

$u: A \to B$ such that

$u(x) = y$ is one - to - one Then:

$$g(y) = \begin{cases} f(w(y)) \cdot |w'(y)| & , y \in B \\ 0 & , \text{else}. \end{cases}$$

where $w = u^{-1}$

Theorem 4

X_1, X_2 random variables on space A

Y_1, Y_2 random variable on space B

$u : A \to B$ such that

$u(X_1, X_2) = (Y_1, Y_2)$

$Y_1 = u_1(X_1, X_2)$

$Y_2 = u_2(X_1, X_2)$

is one - to - one Then: or $|\det(J)|$

$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) \cdot |J| & ; (y_1, y_2) \in B \\ 0 & , \text{else} \end{cases}$$

where $w_1 = u_1^{-1}$, and $\Rightarrow$

$w_2 = u_2^{-1}$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$ ← Jacobian matrix

Ex 1

let X be a r.v with p.d.f

$$f(x) = \begin{cases} 2x & , 0<x<1 \\ 0 & , \text{else} \end{cases}$$

Find the distribution of $Y = 8X^3$

$A = \{x : 0<x<1\}$

$y = 8x^3$

$B = \{y : 0<y<8\}$

$x = \frac{y^{\frac{1}{3}}}{2} = w(y)$

$w'(y) = \frac{dx}{dy} = \frac{1}{3} \frac{y^{\frac{1}{3}-1}}{2} = \frac{1}{6y^{\frac{2}{3}}}$

$$g(y) = \begin{cases} f(w(y))\,|w'(y)| & , y \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} f\left(\frac{y^{\frac{1}{3}}}{2}\right) \left|\frac{1}{6y^{\frac{2}{3}}}\right| & , 0<y<8 \\ 0 & , \text{else} \end{cases}$$

$\Rightarrow$

$$= \begin{cases} 2\left(\frac{y^{\frac{1}{3}}}{2}\right) \cdot \frac{1}{6y^{\frac{2}{3}}} & , 0<y<8 \\ 0 & , \text{else} \end{cases}$$

$$\boxed{g(y) = \begin{cases} \frac{1}{6y^{\frac{1}{3}}} & , 0<y<8 \\ 0 & , \text{else} \end{cases}}$$

Ex :

معناه $\begin{cases} X_1, X_2 \text{ random sample } \sim U[0,1] \\ X_1, X_2 \text{ i.i.d } \sim U[0,1] \end{cases}$

Transformation $\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_1 - X_2 \end{cases}$

Find marginals of Y_1, Y_2

$$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2)) \cdot |\det(J)|$$

$$f_1(x_1) = \begin{cases} 1 & , 0<x_1<1 \\ 0 & , \text{else} \end{cases}$$

$$f_2(x_2) = \begin{cases} 1 & , 0<x_2<1 \\ 0 & , \text{else} \end{cases}$$

$\Rightarrow$

X_1, X_2 indep

$$f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

$$= \begin{cases} 0 , & 0<x_1<1 \\ 0 , & \text{else} \end{cases} \cdot \begin{cases} 1 , & 0<x_2<1 \\ 0 , & \text{else} \end{cases}$$

$$f(x_1, x_2) = \begin{cases} 1 , & 0<x_1<1 \ , \ 0<x_2<1 \\ 0 , & \text{else} . \end{cases}$$

$$\begin{cases} y_1 = x_1 + x_2 = u_1(x_1, x_2) \\ y_2 = x_1 - x_2 = u_2(x_1, x_2) \end{cases} \Leftrightarrow \begin{bmatrix} x_1 = \dfrac{y_1 + y_2}{2} = w_1(y_1, y_2) \\ x_2 = \dfrac{y_1 - y_2}{2} = w_2(y_1, y_2) \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

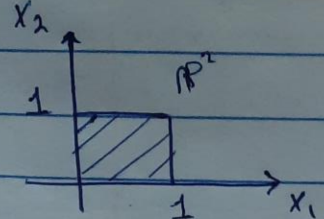
$$\det(J) = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2} \Rightarrow |\det(J)| = \frac{1}{2}$$

$$g(y_1, y_2) = \begin{cases} f\left(\dfrac{y_1+y_2}{2}, \dfrac{y_1-y_2}{2}\right) \cdot \dfrac{1}{2} , & (y_1, y_2) \in B \\ 0 , & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2} , & (y_1, y_2) \in B \\ 0 , & \text{else} \end{cases}$$

$\rightarrow$

$$A = \{(x_1, x_2) : 0 < x_1 < 1 \ , \ 0 < x_2 < 1\}$$



$0 < x_1 < 1$

$0 < x_2 < 1$

$$x_1 = \frac{y_1 + y_2}{2}$$

$$0 < \frac{y_1 + y_2}{2} < 1$$

$$0 < y_1 + y_2 < 2$$

$$\boxed{-y_1 < y_2 < 2 - y_1}$$

$$x_2 = \frac{y_1 - y_2}{2}$$

$$0 < \frac{y_1 - y_2}{2} < 1$$

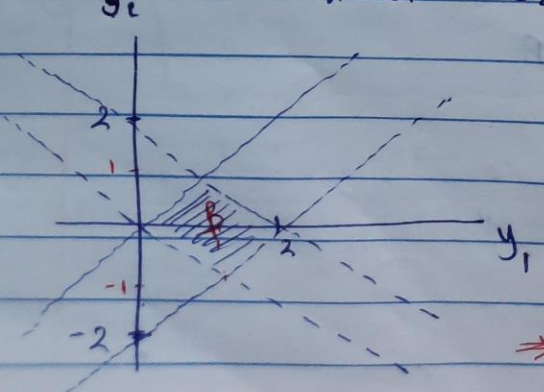
$$0 < y_1 - y_2 < 2$$

$$-2 < y_2 - y_1 < 0$$

$$\boxed{y_1 - 2 < y_2 < y_1}$$

من المتباينات نطلع B

$$B = \{(y_1, y_2) : y_2 > -y_1 \ , \ y_2 < 2 - y_1 \ , \ y_2 > y_1 - 2 \ , \ y_2 < y_1\}$$



Find

$$g_1(y_1) = \begin{cases} y_1 & , 0 \le y_1 \le 1 \\ 2 - y_1 & , 1 \le y_1 \le 2 \\ 0 & , \text{else} \end{cases}$$

$$g_2(y_2) = \begin{cases} 1 + y_2 & , -1 \le y_2 \le 0 \\ 1 - y_2 & , 0 \le y_2 \le 1 \\ 0 & , \text{else} \end{cases}$$

Are Y_1, Y_2 independent? No

$g(y_1, y_2) \neq h_1(y_1) \cdot h_2(y_2)$ for all $(y_1, y_2) \in \mathbb{R}^2$

$g(y_1, y_2) \neq g_1(y_1) \cdot g_2(y_2)$ for all $(y_1, y_2) \in \mathbb{R}^2$

Check the domain in every case!

4.4 Beta, t, F distributions

Beta distribution:

$X_1 \sim \text{Gamma}(\alpha, 1)$

$X_2 \sim \text{Gamma}(\beta, 1)$

X_1, X_2 independent.

Define:

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Result

- $Y_1 \sim \text{Gamma}(\alpha + \beta, 1)$
- $$\boxed{g_2(y_2) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1-y_2)^{\beta-1} & , 0 < y_2 < 1 \\ 0 & , \text{else} \end{cases}}$$

The Beta distribution. p.d.f of

$\Rightarrow$

Note

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for all } \alpha>0,\ \beta>0$$

Note

$$\boxed{Y_2 \sim Beta(\alpha, \beta)}$$

t _distribution

$W \sim N(0, 1)$

$V \sim \chi^2(r)$

W, V independent.

$$T = \frac{W}{\sqrt{\frac{V}{r}}}$$

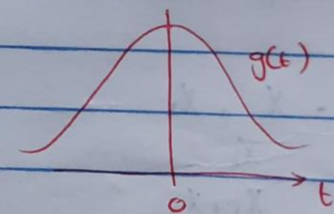
$$U = V$$

Result

$$\boxed{g(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\;\Gamma\left(\frac{r}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}} \quad , t \in \mathbb{R}$$

فقط كي نحفظ

p.d.f of t_dist



F _distribution

$U \sim \chi^2(r_1)$

$V \sim \chi^2(r_2)$

U, V indep.

$$W = \frac{\frac{U}{r_1}}{\frac{V}{r_2}}$$

$$Z = V$$

→ p.d.f of F_dist.

$$\boxed{g(w) = \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)\left(1+\frac{r_1 w}{r_2}\right)^{\left(\frac{r_1+r_2}{2}\right)}}} \quad , w>0$$

$\Longrightarrow$

Notes

① $X_1 \sim Gamma(\alpha, 1)$

$X_2 \sim Gamma(\beta, 1)$

X_1, X_2 indep.

- $X_1 + X_2 \sim Gamma(\alpha + \beta, 1)$
- $\frac{X_1}{X_1 + X_2} \sim Beta(\alpha, \beta)$

$$g(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & else. \end{cases}$$

② $W \sim N(0, 1)$ $\quad$ W, V indep.

$V \sim \chi^2(r)$

$$T = \frac{W}{\sqrt{\frac{V}{r}}}$$

$T \sim$ t-dist with df = r

③ $U \sim \chi^2(r_1)$ $\quad$ U, V indep.

$V \sim \chi^2(r_2)$

$$F = \frac{\frac{U}{r_1}}{\frac{V}{r_2}}$$

$F \sim$ F-dist with $df_1 = r_1$

$df_2 = r_2$

4.5 Extensions of the change-of-variable Techinque

Ex

$X \sim N(0,1)$

$Y = X^2$

Find Dist. of Y

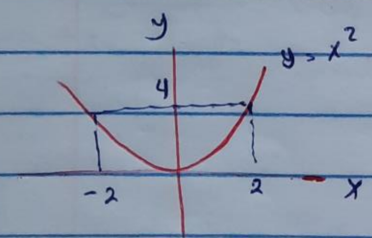
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$

$y = x^2$

$\longleftrightarrow \mathbb{R}$ $\qquad$ $\longleftrightarrow B$

A $\qquad$ 0

$A = \mathbb{R}$ $\qquad$ $B = [0, \infty)$



$X = \sqrt{y}, \quad X \geq 0 \Rightarrow w_1(y) = \sqrt{y} \Rightarrow w_1'(y) = \frac{1}{2\sqrt{y}}$

$X = -\sqrt{y}, \quad X < 0 \Rightarrow w_2(y) = -\sqrt{y} \Rightarrow w_2'(y) = -\frac{1}{2\sqrt{y}}$

$$g(y) = f(w_1(y))\,|w_1'(y)| + f(w_2(y))\,|w_2'(y)|$$

$$= f(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + f(-\sqrt{y})\left|\frac{-1}{2\sqrt{y}}\right|$$

$$f(\sqrt{y})\frac{1}{2\sqrt{y}} + f(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}}$$

$\Rightarrow$

$$g(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y} y^{-\frac{1}{2}} & , y > 0 \\ 0 & , \text{else} \end{cases}$$

$Y \sim \text{Gamma}\left(\frac{1}{2}, 2\right) = \chi^2(1)$

4.6 Distributions of order statistics

Ex: X_1, X_2, X_3, X_4 iid random variables with p.d.f

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{else} \end{cases}$$

Order statistic

Y_1, Y_2, Y_3, Y_4 $\qquad Y_1 < Y_2 < Y_3 < Y_4$

$Y_1 = \min(X_i)$

$Y_2 = 2^{nd}$ order statistic (X_i)

$Y_3 = 3^{rd}$ order statistic (X_i)

$Y_4 = \max(X_i)$

- $g_i(y_i)$: p.d.f of the i^{th} order statistic
- $g_{i,j}(y_i, y_j)$: joint p.d.f i^{th} and j^{th} order statistic

Theorem:

$X_1, \ldots, X_n$ i.i.d random variables with p.d.f f and CDF F, defined on $[a,b]$

let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistic Then:

$\Rightarrow$

① $g_k(y_k) = \frac{n!}{(k-1)!(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left[1 - F(y_k)\right]^{n-k} f(y_k) \quad , a < y_k < b$

② $g_{i,j}(y_i, y_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \left[F(y_i)\right]^{i-1} \left[F(y_j) - F(y_i)\right]^{j-i-1} \left[1 - F(y_j)\right]^{n-j} \cdot f(y_i) f(y_j)$

$, a < y_i < y_j < b$

Ex

① find the p.d.f of max (X_i)

$X_4 = \max(X_i)$

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{else} \end{cases}$$

$a = 0$

$b = 1$

$k = 4$

$n = 4$

$$F(x) = \begin{cases} 0 & , x < 0 \\ \int_0^x 2t\,dt = x^2 & , 0 \leq x < 1 \\ 1 & , 1 \leq x \end{cases}$$

$g_4(y_4) = \frac{4!}{3!\,0!} \left(F(y_4)\right)^3 \left(1 - F(y_4)\right)^0 f(y_4) \quad , 0 < y_4 < 1$

$= 4 \left(F(y_4)\right)^3 f(y_4) \quad , 0 < y_4 < 1$

$$g_4(y_4) = \begin{cases} 4\left((y_4)^2\right)^3 \cdot (2y_4) & , 0 < y_4 < 1 \\ 0 & , \text{else} \end{cases}$$

$\Rightarrow$

$$= \begin{cases} 8y_4^7 & , 0<y_4<1 \\ 0 & , \text{else} \end{cases}$$

Check

$g_1(y_1)$

$g_2(y_2)$

$g_3(y_3)$

② $pr(Y_4 > 0.8) = \int_{0.8}^{\infty} g_4(y_4)\, dy_4$

$$= \int_{0.8}^{1} 8y_4^7\, dy_4 = y_4^8 \Big|_{0.8}^{1} = 1 - (0.8)^8$$

$$= 0.8322$$

Remark

* Ch 1 — section 4.1

  ⇝ CDF method

* section 4.2 – 4.5

  ⇝ change - of - variable method.

* section 4.7

  → m.g.f method.

4.7 The moment generating function Technique

Ex:

$X_1 \sim N(\mu, \sigma^2)$

$X_2 \sim N(\mu, \sigma^2)$

X_1, X_2 independent.

Find the Dist of $X_1 + X_2$

$Y = X_1 + X_2$

$M_Y(t) = E(e^{tY})$

$= E(e^{t(X_1+X_2)})$

X_1, X_2 are independent $= E(e^{tX_1} \cdot e^{tX_2})$

$= E(e^{tX_1}) \cdot E(e^{tX_2})$

$M_{X_1}(t) \cdot M_{X_2}(t)$

$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad (t \in \mathbb{R})$

$\boxed{M_Y(t) = e^{(2\mu)t + \frac{1}{2}(2\sigma^2)t^2}, \quad t \in \mathbb{R}}$

$Y \sim N(2\mu, 2\sigma^2)$

Ex: $X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$ $\rightarrow X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

X_1, X_2 indep $\qquad X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$\Rightarrow$

Remark

$X_1, \ldots, X_n$ independent $N(\mu_i, \sigma_i^2)$

$$Y = \sum_{i=1}^{n} k_i X_i \Rightarrow Y \sim N\left(\sum_{i=1}^{n} k_i \mu_i \, , \, \sum_{i=1}^{n} k_i^2 \sigma_i^2\right)$$

$\boxed{\text{independen} \quad \text{special function}}$

Remark

$X_1, \ldots, X_n$ random sample from $N(\mu, \sigma^2)$

$$Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \Rightarrow Y \sim \chi^2(n)$$

Remark

$X_1, \ldots, X_n$ independent $\chi^2(r_i)$

$$Y = \sum_{i=1}^{n} X_i \rightarrow Y \sim \chi^2\left(\sum_{i=1}^{n} r_i\right)$$

Answer:

$$M_Y(t) = E\left(e^{tX_1 + tX_2 + \cdots + tX_n}\right)$$

$$= E\left(e^{tX_1}\right) \cdots E\left(e^{tX_n}\right)$$

$$= M_1(t) \cdots M_n(t)$$

$$= \frac{1}{(1-2t)^{\frac{r_1}{2}}} \cdots \frac{1}{(1-2t)^{\frac{r_n}{2}}} \, , \quad t < \frac{1}{2}$$

$$= \frac{1}{(1-2t)^{\frac{r_1 + \cdots + r_n}{2}}} \, , \quad t < \frac{1}{2}$$

$\rightarrow$

4.8 The Dist of $\bar{X}$ and $\frac{nS^2}{\sigma^2}$

$X_1, X_2, \ldots, X_n$ random sample from $N(\mu, \sigma^2)$

① $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

② $\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

③ $\bar{X}$ and S^2 are independent

where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

4.9 Expectations of functions of random variable.

$X_1, X_2, \ldots, X_n$ random variables with means $\mu_1, \mu_2, \ldots, \mu_n$
& Variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

Define $Y = \sum_{i=1}^{n} K_i X_i$

① $E(y) = \mu_y = \sum_{i=1}^{n} k_i \mu_i$

② $var(Y) = \sigma_y^2 = \sum_{i=1}^{n} k_i \sigma_i^2 + \begin{cases} 0 & , X_1, \ldots X_n \text{ indep} \\ \sum_{j=1}^{n}\sum_{\substack{i=1 \\ i \neq j}}^{n} k_i k_j \rho_{ij} \sigma_i \sigma_j & , X_1, \ldots X_n \text{ not indep} \end{cases}$

Note $\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i \neq j}}^{n} k_i k_j \rho_{ij} \sigma_i \sigma_j = 2\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i<j}}^{n} k_i k_j \rho_{ij} \sigma_i \sigma_j$